Introduction to G_2-structures (RPD, 06.01.2026)

References:

- Dominic D. Joyce: [Joy00]
  "Compact Manifolds with Special Holonomy"
- Dietmar A. Salamon, Thomas Walpuski: [SW17]
  "Notes on the Octonions"
- Spiro Karigiannis: [Kar05]
  "Flows of G_2-structures"
- Simon Salamon: [Sal85]
  "Riemannian Geometry and Holonomy Groups"
- Robert L. Bryant:
  "Some remarks on G_2-structures" [Bry25]
  "Metrics with exceptional holonomy" [Bry87]
  (and some nice comments on Math overflow...)

Small note: My G ($=\mathcal{G}$) looks like a S ($=S$). Do not get confused, we talk about G_2-structures.

Overview:

1) The group G_2
2) Relations to the Octonions
3) G_2-structure
4) Metric and orientation
5) Decomposition of forms
6) Torsion-free G_2-structures

1. The group G_2 [Joy00], [SW17]

Consider $\mathbb{R}^7$ with an action of $GL(7,\mathbb{R}) = GL(\mathbb{R}^7)$. On $\mathbb{R}^7$, have coordinates $(x_1, \dots, x_7)$. We define a 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$ on $\mathbb{R}^7$ via $dx_{ijk} = dx_i \wedge dx_j \wedge dx_k$ as:

$$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$$

↳ [Joy00]

We denote the group of its automorphisms by:

$$G_2 := \{ g \in GL(\mathbb{R}^7) \mid g^* \varphi_0 = \varphi_0 \},$$

as a subgroup of $GL(7,\mathbb{R})$.

We list some properties without proving them. For a proof, we refer to [SW17, Thm. 8.1].

<u>Thm.</u> (properties of G_2)

G_2 is a Lie group with properties:

- 14-dimensional
- (semi-) simple
- connected
- simply connected
- closed $\}$ compact
- $\subset SO(7)$ $\}$ compact
- ⇝ "independent of φ_0" (later)

2. Relation to the Octonions [SW17], [Bry25]

Def. (Cross product)

V f.d. real Hilbert space. (that is, a complete normed space where the inner product is usually denoted by $\langle \cdot , \cdot \rangle$). A skew symmetric bilinear pairing

$$\times : V \oplus V \longrightarrow V, \quad (u,v) \longmapsto u \times v$$

is called <u>cross-product</u> if

$$\left.\begin{array}{l} \text{i) } \langle u \times v, u \rangle = \langle u \times v, v \rangle = 0 \\ \text{ii) } |u \times v|^2 = |u|^2 |v|^2 - \langle u, v \rangle^2 \end{array}\right\} \quad u \times u = 0$$

Rem. The condition that i) holds $\forall u, v \in V$ is equivalent to the existence of an alternating 3-form $\phi : V^3 \longrightarrow \mathbb{R}$

$$\phi(u,v,w) := \langle u \times v, w \rangle,$$

called the associative calibration of $(V, \underset{\text{cross product}}{\times})$.

Theo. (Existence of cross products)

V (as above) admits a cross product iff it has dimension:

$$\left.\begin{array}{l} 0 \rightsquigarrow \mathbb{R} \\ 1 \rightsquigarrow \mathbb{C} \end{array}\right\} \text{cross product vanishes}$$

$$3 \rightsquigarrow \mathbb{H} \quad \} \text{ unique up to sign}$$

$$7 \rightsquigarrow \mathbb{O} \quad \} \text{ unique up to orthogonal isomorphism.}^{(*)}$$

The automorphism group of ϕ will be denoted by
$S(V,\phi) := \{ g \in SL(V) : g^* \phi = \phi \}$.
By definition of ϕ, we can similarly say
$S(V,\phi) = \{ g \in GL(V) : gu \times gv = g(u \times v) \ \forall u, v \in V \}$
This will be useful to understand another definition of S_2.

The cross product on $\mathbb{R}^0, \mathbb{R}^1, \mathbb{R}^3$ and $\mathbb{R}^7$ are induced by all the finite normed division algebras (→ see Hurwitz-theorem) $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, respectively. For reasons of accessibility, we will mostly look at the quaternions $\mathbb{H}$ and then discuss what changes for the octonions.
Our goal is to motivate: $S_2 = \mathrm{Aut}(\mathbb{O})$.

The <u>quaternion algebra</u> $\mathbb{H}$ is generated by i, j s.t. $i^2 = j^2 = -1$ and $ij = -ji$. The product ij is linearly independent of i and j s.t. $\dim \mathbb{H} = 4$. For $x \in \mathbb{H}$, have $x = x_0 \cdot 1 + x_1 \cdot i + x_2 \cdot j + x_3 \cdot ij$
In analogy to $\mathbb{C}$, define real and imaginary part to be
$Re(x) = x_0$
$Im(x) = (x_1, x_2, x_3)$.

In order to define a cross product $\times$ on $\mathbb{R}^3$, we identify $\mathbb{R}^3 \simeq Im(\mathbb{H})$.

Then

$$\times : \mathbb{R}^3 \oplus \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto u \times v := \operatorname{Im}(u \cdot v).$$

Up to sign, this coincides with the familiar definition of the cross product:

$$(u_1 i + u_2 j + u_3 ij)(v_1 i + v_2 j + v_3 ij)$$
$$= -u_1 v_1 - u_2 v_2 - u_3 v_3$$
$$+ (u_2 v_3 - u_3 v_2) i + (u_3 v_1 - u_1 v_3) j$$
$$+ (u_1 v_2 - u_2 v_1) ij$$

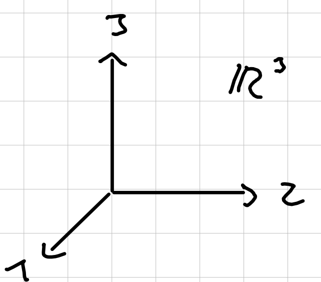
s.t. $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$ as usual.

The associated 3-form $\langle u \times v, w \rangle$ is given by

$$\phi(u,v,w) = dx_1 \wedge dx_2 \wedge dx_3 =: dx_{123}$$

(since:

$$\langle u \times v, w \rangle = (231) - (321) + (312) - (132) + (123) - (213$$
$$= (123) - (132) + (231) - (213) + (312) - (321$$
$$= 1 \wedge 2 \wedge 3 \,)$$



This gives the standard structure (and orientation) of $\mathbb{R}^3$.

Any automorphism of the quaternions mixes the imaginary part of the orthonormal basis $\{1, i, j, ij\}$. Hence, $\operatorname{Aut}(\mathbb{H}) \simeq SO(3)$, which also leaves dx_{123}

invariant. As the cross product inherits the structure of $\mathrm{Im}(\mathbb{H})$, have $S(\mathbb{R}^3, \phi = \pm \mathrm{vol}) \simeq SO(3)$.

For $\mathbb{R}$ get a trivial cross product from $\mathbb{R} \simeq \mathrm{Im}(\mathbb{C})$.
$\mathrm{Aut}(\mathbb{C}) \simeq \mathbb{Z}_2$, since any automorphism of $\mathbb{C}$ is either $\mathrm{id}_{\mathbb{C}}$ or complex conjugation. $\mathrm{Aut}(\mathbb{R}) = 1$

Similarly, $\mathbb{R}^7 \simeq \mathrm{Im}(\mathbb{O})$. The octonions have 3 generators $\{i, j, k\}$, which are anticommuting and square to 1.
$\mathbb{O}$ is also non-associative by consistency:

$$-k(ij) = (ij)k = -i(jk) = i(kj) = -(ik)j = (ki)j = -k(ij)$$

Depending on how the basis is chosen, different cross products and associated forms can arise:

[Joy00]

$$\phi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$$

[SW17]:

$$\phi_0' = dx_{123} - dx_{145} - dx_{167} - dx_{246} + dx_{257} - dx_{347} - dx_{356}$$

arises from different Octonion structures:

Coord. x_i	1	2	3	4	5	6	7
[Joy00]	i	j	ij	k	ik	jk	$i(jk)$
[SW17]	i	j	ij	k	ki	kj	$\overset{\shortparallel}{k(ij)}$

The associated cross products are ugly. For [SW17], have e.g. $(u \times v)_1 = u_2 v_3 - u_3 v_2 - u_4 v_5 + u_5 v_4 - u_6 v_7 + u_7 v_6$.

One would assume that $\mathrm{Aut}(\mathbb{O}) \rightsquigarrow SO(7)$, but this is "too big". Since

$$\begin{aligned} G_2 &:= G(\mathbb{R}^7, \varphi_0) \\ &:= \{g \in SL(7,\mathbb{R}) : g^*\varphi_0 = \varphi_0\} \\ &= \{g \in SL(7,\mathbb{R}) : (gu \times gv) = g(u \times v)\ \forall u,v \in \mathbb{R}^7\} \\ \underset{\text{non-trivial}}{\simeq} &\{g \in SL(7,\mathbb{R}) : g(u)\cdot g(v) = g(uv)\ \forall u,v \in \mathrm{Im}(\mathbb{O})\} \\ &= \mathrm{Aut}(\mathbb{O}). \end{aligned}$$

it is at least intuitive that the automorphism group of the octonions should be G_2.

More precisely, $\mathbb{O}$ has more restricting structure compared to $\mathbb{H}$. Any $g \in \mathrm{Aut}(\mathbb{H})$ acts transitively on orthonormal pairs in $\mathrm{Im}(\mathbb{H}) \rightsquigarrow SO(3)$. But, any $g \in \mathrm{Aut}(\mathbb{O})$ that fixes three elements of $\{i, j, ij, k, ik, jk, (ij)k\}$ is $\mathrm{id}_{\mathbb{O}}$ already and any orthonormal triple is mapped uniquely to any other orthonormal triple by some $g \in \mathrm{Aut}(\mathbb{O})$ $\rightsquigarrow SO(7)$ is too big (for more, look at Stiefel mfds).

The fact that any orthonormal triple in $\mathrm{Im}(\mathbb{O})$ can be mapped to any other orthonormal triple motivates that our definition of G_2 does not depend on the explicit 3-form ϕ_0, but rather on the fact that the 3 form arises from the octonion structure.

<u>Thm.</u> For $\phi, \phi' \in \Lambda^3(V)^*$ non-degenerate (for u, v lin. independent $\exists\, w \in V$ st. $\phi(u,v,w) \neq 0$) and
- u, v, w orthonormal st. $\phi(u,v,w) = 0$
- u', v', w' orthonormal st. $\phi'(u',v',w') = 0$

$\exists\, g \in \mathrm{Aut}(V)$ st. $g^*\phi' = \phi$ and $g(u,v,w) = (u',v',w')$

$\rightsquigarrow$ $G_2 \simeq G(\mathbb{R}^7, \phi)$ for any sufficient ϕ.

We should also note that any ϕ is non-degenerate iff it admits a compatible inner product i_ϕ (which then is also uniquely determined by ϕ). The inner product i is a map $i : V \times \Lambda^3 V^* \longrightarrow \Lambda^2 V^*$, where $(u, \phi) \longmapsto i_\phi(u)\phi = \phi(u, \cdot, \cdot)$, i.e. $(i_\phi(u)\phi)_{jk} = \alpha_{ijk} u^i$
In differential geometry, usually have: $i(v)\phi \equiv v \lrcorner \phi$
Moreover, ϕ orients V as follows:

<u>Prop.</u> V 7-dim. real Hilbert and $\phi \in \Lambda^3 V^*$. If ϕ is non-degenerate, then there is an orientation of V st. the associated volume form $\mathrm{vol} \in \Lambda^7 V^*$ satisfies $i_\phi(u)\phi \wedge i_\phi(v)\phi \wedge \phi = 6\langle u, v\rangle \mathrm{vol}$ $\forall\, u, v \in V$

3. G_2 - structures [Joy00], [Kar08]

First, some aspects about special fibre bundles.

Def. (principle bundle)

Let M be a smooth mfd. and G a Lie group.
A principle bundle $P \xrightarrow{\pi} M$ is a smooth mfd. P with a smooth projection $\pi : P \longrightarrow M$. Additionally, there is a smooth and free G-action on P, $q \xmapsto{g} g \cdot q$ for $g \in G$ and $q \in P$. The projection $\pi : P \longrightarrow M$ is a fibration whose fibres are the G-orbits of $q \in P$, i.e. $\pi^{-1}(p) \simeq Gq \simeq G$ for $\pi(q) = p \in M$.

As an example:
Consider $\pi : S^1 \longrightarrow S^1$ with $\pi(z) = z^2$. This a principle bundle with fibre $\mathbb{Z}_2$, seen as $(\{\pm 1\}, \cdot)$.

Def. (frame bundle)

M mfd. and $TM \xrightarrow{\pi} M$, $T_pM \simeq \mathbb{R}^n$. Define
$F := \{(p, \partial_1, \dots, \partial_n), m \in M \text{ and } (\partial_1, \dots, \partial_n) \text{ basis for } T_pM\}$
We set:

- $\pi : F \longrightarrow M$, $\pi(p, \partial_1, \dots, \partial_n) = p$
- for $A \in GL(n, \mathbb{R})$, define $A(p, \partial_1, \dots, \partial_n) = (p, \partial_1', \dots, \partial_n')$ with $\partial_i' = A_{ij}\partial_j$.

Thus, F is a principle bundle with fibre $GL(n, \mathbb{R})$, called the frame bundle.

Each point in the fibre of F determines an isomorphism between T_pM and $\mathbb{R}^k$ via a change of basis, i.e. $u \in F|_p : T_pM \xrightarrow{\sim} \mathbb{R}^k$ (where we identify $T_pM \xrightarrow{A} T_pM$ with $T_pM \xrightarrow{A} T_pM \xrightarrow{f} \mathbb{R}^k$, with $(\partial_i) = e_i$, and call $u = f \circ A$).

$$\begin{tikzcd} T_pM \arrow[rr, "A"] \arrow[dr, "u"'] & & T_pM \arrow[dl] \\ & \mathbb{R}^k & \end{tikzcd} \quad \text{and} \quad F|_p \ni A \longleftrightarrow u \in F|_p$$

__Def.__ (G-structure)

M^n mfd. and F corresponding frame bundle, i.e. F is principle bundle with fibre $GL(n,\mathbb{R})$. Let G be Lie subgroup of $GL(n,\mathbb{R})$. Then a __G-structure__ is a principle subbundle P^G of F.

A G_2-structure then is a principle subbundle of F with fibre G_2.

__Goal:__ G_2 structure $\longleftrightarrow$ positive 3-form on M

From now on, M is a 7-dimensional manifold.

__Def.__ (positive 3-form)

A $\varphi \in \Lambda^3 T_p^*M$ is said to be positive if there exists an orientation preserving isomorphism $u: T_pM \xrightarrow{\sim} \mathbb{R}^7$ such that $u^*\phi_0 = \varphi$ (i.e. the isomorphism identifies $\varphi \in \Lambda^3 T_p^*M$ and $\phi_0 \in \Lambda^3(\mathbb{R}^7)^*$ from before).

We denote the set of all positive 3-forms by $(\Lambda^3_+ M)_p$ (alternatively $\mathcal{P}^3_p M$), and get $\Omega^3_+(M) := \bigsqcup_{p\in M} (\Lambda^3_+ M)_p$ naturally (alternatively $\mathcal{P}^3 M$).

We have that $\Omega^3_+(M) \subset \Omega^3(M)$ is an open subset, since: $\dim \Omega^3_+(M) = \dim \left({}^{SL(7,\mathbb{R})}/_{G_2} \right) = 7^2 - 14 = 35$

$$\dim \Omega^3(M) = \binom{7}{3} = 35.$$

<u>3-form $\varphi \rightsquigarrow G_2$-structure</u> (implication is unique)

Given a $\varphi_0 \in \Omega^3_+(M)$, we define

$F_\varphi := \{ u_p \in \mathrm{Hom}(T_pM, \mathbb{R}^7) : p \in M \text{ and } u_p^* \phi_0 = \varphi_0|_p \} \subset F$

This is a subbundle of the frame bundle F whose fibres are maps $u_p : T_pM \longrightarrow \mathbb{R}^7$ that map ϕ_0 to φ_0. Thus, u respects the G_2-invariance of ϕ_0 and turns F_φ into a G_2-structure.

<u>G_2-structure $\rightsquigarrow$ 3-form φ:</u>

Given a G_2-structure on M, we can define a 3-form on M via an isomorphism $u_p : T_pM \longrightarrow \mathbb{R}^7$, $u_p^* \phi_0 = \varphi_0|_p$. However, φ_0 is not necessarily in $\Omega^3_+(M)$. We have to require in addition that u_p is orientation preserving. The orientation of M is thus induced from the G_2-structure (see below for the metric and vol in detail).

Thus, have: $\varphi \in \Omega^3_+(M) \xleftrightarrow{1:1} G_2$-structure

4. Metric and orientation [Ka 05]

As is clear from the aforegoing discussion, a G_2 manifold is orientable. We will also look into this more deeply here.

Before that, one might wonder whether any 7-mfd. admits a G_2-structure. No, but the conditions can be stated.

<u>Def.</u> (spin structure)

Let M be smooth mfd. and $P^{SO(n)}$ an $SO(n)$-structure (this is unique for a given Riemannian metric and orientation). $SO(n)$ has double cover $Spin(n)$, which is cpt., connected and simply connected.

A spin-structure $\tilde{P}$ on M is a principal bundle $P^{Spin(n)} = \tilde{P}$ on M with fibre $Spin(n)$ and a bundle map $\pi : \tilde{P} \longrightarrow P$, that may locally be regarded as the double cover $\pi : Spin(n) \longrightarrow SO(n)$.

<u>Theo.</u> A 7-mfd. M admits a G_2-structure iff it is orientable and spin.

[for the pro's, this is equivalent to the vanishing of the first and second Stiefel-Whitney classes $w_1(M) = 0 = w_2(M)$].

From now on, we will not distinguish between a G_2-structure and the associated 3-form φ, for the sake that they are in 1:1-correspondence.

Famously, any n-mfd. admits an orientation iff there exists a smooth n-form. For Riemannian mfds., this is the volume form vol. For a G_2-structure, both is determined by φ.

On vector spaces, we get an orientation from ϕ via

$$(i(u)\phi) \wedge (i(v)\phi) \wedge \phi = 6 \langle u, v \rangle \, vol \quad \forall u, v \in V.$$

We mimic this non-linear relation for the G_2-structure φ, where we switch: $i(v)\phi \rightsquigarrow v \lrcorner \varphi$. Explicitly:

$$(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6 \, g_\varphi(X, Y) \, vol_\varphi \quad \forall X, Y \in \mathfrak{X}(M)$$

We will simplify $g \equiv g_\varphi$ and $vol \equiv vol_\varphi$.

When we choose $X = \partial_i$, $Y = \partial_j$, the LHS reads

$$(\partial_i \lrcorner \varphi) \wedge (\partial_j \lrcorner \varphi) \wedge \varphi = B_{ij} \, dx^1 \wedge \dots \wedge dx^7$$

for some $B_{ij} = B_{ji}$, $B \in \Gamma(Sym^2(T^*M) \otimes \Lambda^7 T^*M)$. Thus

$$B_{ij} \, dx^1 \wedge \dots \wedge dx^7 = -6 \, g(\partial_i, \partial_j) \, vol = -6 \, g_{ij} \, vol$$

For a Riemannian mfd, have: $vol = \sqrt{\det(g)} \, dx^1 \wedge \dots \wedge dx^7$

st. get:

$$B_{ij} = -6 \, g_{ij} \sqrt{\det(g)}$$

(since $dx^1 \wedge \dots \wedge dx^7 (\partial_1, \dots, \partial_7) = 1$ by construction)

$$\Rightarrow \det(B) = (-6)^7 \det(g)^{\frac{7}{2}} \det(g) = -6^7 \det(g)^{\frac{9}{2}}$$

$$\Rightarrow \sqrt{\det(g)} = -\frac{1}{6^{\frac{7}{9}}} \det(B)^{\frac{1}{9}}$$

and thereby get:

- $g_{ij} = -\frac{1}{6} \frac{B_{ij}}{\sqrt{\det(g)}} = \frac{1}{6^{\frac{2}{3}}} \frac{B_{ij}}{\det(B)^{\frac{1}{9}}}$

- $\mathrm{vol} = -\frac{1}{6^{\frac{7}{3}}} \det(B)^{\frac{1}{3}} dx^1 \wedge \ldots \wedge dx^7$

Some authors use $+6$ instead of -6, which results in the same metric but gives opposite orientation.

A change of basis of $d\tilde{x}^i = P^i_j \, dx^j$ results in
$\tilde{B}_{ij} = P_i^a P_j^b \det(P) B_{ab}$, since
$B \in \Gamma(\mathrm{Sym}^2(T^*M) \otimes \Lambda^7 T^*M)$. Then:

$$\tilde{g}_{ij} = -\frac{1}{6^{\frac{2}{3}}} \frac{P_i^a P_j^b \det(P) B_{ab}}{\left(\det(P) \det(P)^7 \det(B)\right)^{\frac{1}{9}}}$$

$$= -\frac{1}{6^{\frac{2}{9}}} P_i^a P_j^b \frac{B_{ab}}{\det(B)^{\frac{1}{9}}}$$

$$= P_i^a P_j^b g_{ab}$$

5. Decomposition of forms [Kar 05]

On any oriented Riemannian n-mfd., we can define an operator $*: \Lambda^k T^*M \longrightarrow \Lambda^{n-k} T^*M$, called the Hodge-star-operator (or -map, when seen as a bundle homomorphism), that satisfies

$\omega \wedge * \eta = \langle \omega, \eta \rangle_g \mathrm{vol}$.

$\hookrightarrow$ $\langle \omega, \eta \rangle_g := \det(\langle (\omega^i)^\#, (\eta^j)^\# \rangle)$

$\hookrightarrow$ $(\omega^i)^\# = g^{ij} \omega_j$, i.e. $\omega^\# = g^{ij} \omega_j \partial_i$, and

$\langle \cdot, \cdot \rangle_g = g(\cdot, \cdot)$ for vector fields

The Hodge $*$ is (up to signs) the "what is missing in the form" operator. In our case of the G_2-structure [Joy 00]

$\varphi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356}$

have

$*\varphi_0 = dx_{4567} + dx_{2367} + dx_{2345} + dx_{1357} - dx_{1346} - dx_{1256} - dx_{1247}$

As it depends on g and vol, which depend on φ, it is clear that a G_2-structure induces a Hodge $*$ in a non-linear way (although $*$ is a linear operator).

On a mfd. that admits an almost complex structure, the (co-) tangent bundle decomposes as a Whitney sum into holomorphic and anti-holomorphic part. This carries over to the forms, such that the span of k-forms decomposes into (p,q)-forms with holomorphic (p)

and anti-holomorphic (q) part, i.e.

$$\Lambda^k_{\mathbb{C}} M = \bigoplus_{p+q=k} \Lambda^{p,q} M .$$

A G_2-structure induces a similar construction.

<u>Prop.</u> Let M be 7-mfd. with a G_2-structure (φ, g_φ). Then $\Omega^k(M)$ splits orthogonally into components $\Omega^k_\ell(M)$ of (pointwise)$^{(*)}$ dimension ℓ, which are irreducible representations of G_2 of dimension ℓ.

0) $\Omega^0(M) = C^0(M)$ (continuous $\mathbb{R}$-functions, irreducible)

i) $\Omega^1(M) = \Omega^1_7(M)$ (irreducible)

ii) $\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M)$

iii) $\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M)$

iv) $\Omega^4(M) = \Omega^4_1(M) \oplus \Omega^4_7(M) \oplus \Omega^4_{27}(M)$

v) $\Omega^5(M) = \Omega^5_7(M) \oplus \Omega^5_{14}(M)$

vi) $\Omega^6(M) = \Omega^6_7(M)$ (irreducible)

vii) $\Omega^7(M) = \Omega^7_1(M)$ (irreducible)

Note: $\Omega^{7-k}(M) \simeq * \Omega^k(M)$, where $\simeq$ is an isometry.

Explicitely, the spaces look like this:

- $\Omega^2_7(M) = \{ (X \lrcorner \varphi) \mid X \in \mathfrak{X}(M) \}$
  $= \{ \beta \in \Omega^2(M) \mid *(\varphi \wedge \beta) = -2\beta \}$
- $\Omega^2_{14}(M) = \{ \beta \in \Omega^2(M) \mid \beta \wedge * \varphi = 0 \}$
  $= \{ \beta \in \Omega^2(M) \mid *(\varphi \wedge \beta) = \beta \}$

- $\Omega^3_1(M) = \{ f\varphi \mid f \in C^\infty(M) \}$
- $\Omega^3_7(M) = \{ (X \lrcorner * \varphi) \mid X \in \mathfrak{X}(M) \}$
- $\Omega^3_{27}(M) = \{ \gamma \in \Omega^3(M) \mid \gamma \wedge \varphi = 0 = \gamma \wedge * \varphi \}$

$$= \{ h_{ij}\, dx^i \wedge (\partial_j \lrcorner \varphi) \mid h_{ij} = h_{ji},\ g^{ij} h_{ij} = 0 \}$$

For details and explicit derivations, we refer to [Ka 05], chapter 2.2.

The prefactor of $(X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi = -6\, g(X,Y)\, \mathrm{vol}$, the ∓ 6, determines the sign conventions in Ω^2_7 and Ω^2_{14}. Have:

	Ω^2_7	Ω^2_{14}
-6	$*(\varphi \wedge \beta) = -2\beta$	$*(\varphi \wedge \beta) = \beta$
$+6$	$*(\varphi \wedge \beta) = +2\beta$	$*(\varphi \wedge \beta) = -\beta$

6. Torsion free G_2-structures [Joy00]

As we have a 7 mfd. with a Riemannian metric, the Fundamental theorem of Riemannian geometry asserts the existence of a Levi-Civita connection on M, which we will denote by ∇ (this is unique).

Def. (torsion, torsion free)
Let M be 7-mfd., (φ, g) a G_2 structure on M and ∇ the Levi-Civita connection of g. We call $\nabla\varphi$ the torsion of the G_2-structure. (φ, g) is called torsion free if $\nabla\varphi = 0$.

Torsion-freeness is a far from trivial condition, as it is a non-linear partial differential equation on φ. Recall that:

$$\begin{tikzcd} & & \nabla \\ \varphi \arrow[r, rightsquigarrow] & g \arrow[ur, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\ & & * \end{tikzcd}$$

i.e. ∇ depends on φ non-linearly.

For reasons that will become clear in a moment, we refer to a triple (M, φ, g) with

- M (smooth) 7-mfd.
- (φ, g) torsion-free G_2-structure

as a G_2-manifold.

Alternative (but equivalent) definitions can be given via the following proposition.

Prop. M 7-mfd. and (φ, g) G_2-structure. Then are equivalent:

i) (φ, g) is torsion free

ii) $\nabla\varphi = 0$ with ∇ Levi-Civita connection

iii) $\mathrm{Hol}(g) \subseteq G_2$ and φ is the induced 3-form.

iv) $d\varphi = d*\varphi = 0$ on M.

Note that $*$ also depends on φ s.t. the condition $d*\varphi = 0$ is also a non-linear PDE.

As we will see soon, G_2-manifolds can be constructed out of Calabi-Yau manifolds. These are famously Ricci-flat.

Prop. (M^7, g) Riemannian mfd.. Then:

$\mathrm{Hol}(g) \subseteq G_2 \Rightarrow \mathrm{Ric}(g) = 0$ i.e. M is Ricci-flat.

We now want to look at (more or less) explicit examples of G_2-manifolds. Therefore, we need to investigate the condition $\mathrm{Hol}(g) \subseteq G_2$ in more detail.

Thm. (subgroups of G_2)

The only connected Lie subgroups of G_2 which can be the (restricted) holonomy groups of a Riemannian 7-mfd. are $1 \subset SU(2) \subset SU(3) \subset G_2$. Thus, for (M, φ, g) G_2-manifold, then

$Hol^0(g) = 1, SU(2), SU(3), G_2$.

Idea of proof:

We motivated a G_2 structure using the 3-form φ_0 on $\mathbb{R}^7$. $\mathbb{R}^7$ can be decomposed:

- $\mathbb{R}^7 \simeq \mathbb{R}^3 \times \mathbb{C}^2$:

  on $\mathbb{R}^3$, have standard euclidean metric

  $h = dx_1^2 + dx_2^2 + dx_3^2$

  on $\mathbb{C}^2$, have standard $SU(2)$-structure

  $(\mathbb{C}^2, \omega_I, \Omega)$ (hyperkähler I, J, K)

  $\rightsquigarrow \omega_I$: Kähler form wrt. cplx. structure I

  $\rightsquigarrow \Omega = \omega_J + i\omega_K$: holomorphic symplectic form

  A metric on $\mathbb{R}^3 \times \mathbb{C}^2$ is given by $g = h \times g_{\mathbb{C}^2}$ and the defining 3-form can be expressed as

  $\varphi = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega_I + dx_2 \wedge \omega_J + dx_3 \wedge \omega_K$.

  The subgroup of $\mathbb{R}^7$ fixing $\mathbb{R}^3$ is thereby $SU(2)$.

- $\mathbb{R}^7 \simeq \mathbb{R} \times \mathbb{C}^3$ with $\mathbb{R}$ as usual and

  $(\mathbb{C}^3, \omega, \Omega)$ the standard $SU(3)$ structure.

  $\rightsquigarrow \omega$: Kähler form

  $\rightsquigarrow \Omega$: holomorphic volume form

  Have $\varphi = dt \wedge \omega + \mathrm{Re}(\Omega)$ and $g = dx^2 \times g_{\mathbb{C}^3}$.

  The stabilizer of $\mathbb{R}$ is $SU(3)$.

For completeness, also note that for (M, φ, g) compact G_2-mfd, we have that: $Hol(g) = G_2 \Leftrightarrow \pi_1(M)$ is finite

The "other direction" is quite powerful.

<u>Prop.</u> 1) Suppose (Y, g_Y) is a Riemannian 4 mfd. with holonomy $SU(2)$. Then Y admits a

- cplx. structure J
- a Kähler form ω (with $d\omega = 0$)
- and a holomorphic volume form Ω with $d\Omega = 0$.

With the standard metric $h = dx_1^2 + dx_2^2 + dx_3^2$ on $\mathbb{R}^3$, define a

- metric $g = h \times g_Y$ on $\mathbb{R}^3 \times g_Y$
- 3-form:

$$\varphi = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \omega + dx_2 \wedge \mathrm{Re}(\Omega) - dx_3 \wedge \mathrm{Im}(\Omega)$$

Then (φ, g) is a torsion free G_2 - structure and

$$* \varphi = \frac{1}{2} \omega \wedge \omega + dx_2 \wedge dx_3 \wedge \omega - dx_1 \wedge dx_3 \wedge \mathrm{Re}(\Omega) - dx_1 \wedge dx_2 \wedge \mathrm{Im}(\Omega).$$

2) Suppose (X, g_X) is a Riemannian 6 mfd. with holonomy $SU(3)$. Then X admits a

- cplx. structure J
- a Kähler form ω (with $d\omega = 0$)
- and a holomorphic volume form Ω with $d\Omega = 0$.

With $h = dx^2$ on $\mathbb{R}$, define a

- metric $g = dx^2 \times g_Y$
- 3-form: $\varphi = dx \wedge \omega + \mathrm{Re}(\Omega)$.

Then (φ, g) is a torsion free G_2-structure and

$$*\varphi = \frac{1}{2}\omega\wedge\omega - dx \wedge \mathrm{Im}(\Omega).$$

Proof: See [Joy00], Prop. 11.1.1 and Prop. 11.1.2.

Short recap:

- Kähler mfd.:
  Complex mfd. with hermitian metric g (i.e. $g(X,Y) = g(JX, JY)$) and 2-form $\omega(X,Y) := g(JX, Y)$. g is Kähler metric and ω is Kähler form if $d\omega = 0$.
- Calabi-Yau mfd.:
  Compact (connected) Kähler mfd. (M, J, g) with $SU(m)$ holonomy (physics: usually say Ricci-flat)

Explicitly, get torsion free G_2 structures on:

- $T^3 \times K3$, with $K3$ a Calabi-Yau two-fold (it is proven that all $K3$ are diffeomorphic $\rightsquigarrow$ Kodaira. However, there is a 20-parameter family of inequivalent cplx. structures).
  On the torus, one may choose a flat metric. Calabi-Yau metrics on the other hand are notoriously hard to obtain. But Calabi-Yau's can be constructed from complex projective spaces, e.g. the Fermat quartic:
  $FQ \cong \{[z_0, \dots, z_3] \in \mathbb{CP}^3,\ z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\}$
  One obtains a metric on $K3$ by restricting the standard

Fubini-Study metric g_{FS} on $\mathbb{C}P^3$ (which is Kähler) to the algebraic hypersurface FQ: $g_{FS} = \sum g_{i\bar{i}}\, dz^i \otimes d\bar{z}^{\bar{i}}$

$$g_{i\bar{i}} = \frac{1}{2}\left(\frac{\delta_{i\bar{i}}}{1+|z|^2} - \frac{\bar{z}^j z^{\bar{k}}}{(1+|z|^2)^2}\right).$$

- $\mathbb{R} \times CY^6$ or $S^1 \times CY^6$, where CY^6 is any Calabi-Yau 3-fold and $S^1 = \mathbb{R}/\mathbb{Z}$. On S^1 (parametrised as $(\cos(\theta), \sin(\theta))$), get metric $g_{S^1} = d\theta^2$, while $g_{\mathbb{R}} = dx$ as usual. The quintic CY^6 is defined as $Q = \{[z_0, \dots, z_4] \in \mathbb{C}P^4 \mid \sum_{i=0}^{4} z_i^2 = 0\}$

Many background on complex manifold theory is given in John M. Lee: "Introduction to complex manifolds", while there is also enough in [Joy00]. Funny Calabi-Yau stuff appears also frequently in String Theory literature.

The holonomy groups of the manifolds above are given by $SU(2)$ and $SU(3)$, respectively. Joyce describes in [Joy00] how adequate quotients of these spaces (could) give rise to $Hol(g) = G_2$. This is beyond this notes' scope (which is my scope...).